Day 29
Wed Jul 12
Note: the characteristic polynomial of a matrix A is defined by the textbook as

\[ \det(\lambda I - A) \]

We have defined it as

\[ P_A(\lambda) = \det(A - \lambda I) \]

\[ \det(\lambda I - A) = \det(-(A - \lambda I)) \]

\[ = (-1)^n \det(A - \lambda I) \]

We now update the list of non-singular equivalences.

The following statements are all equivalent for a square nxn matrix:

1) A is non-singular (that is, the inverse $A^{-1}$ exists)

2) $Ax = 0$ has only the trivial solution

3) The RREF of A is the identity matrix $I_n$
4) $Ax = b$ has a unique solution \[ x = A^{-1}b \]

5) $\det(A)$ is not zero

6) The rows of $A$ are linearly independent

7) The columns of $A$ are linearly independent

8) Zero is not an eigenvalue

Because, if $\lambda = 0$ is an eigenvalue, then 0 must be a root of the characteristic polynomial of $A$:

\[ p_A(\lambda) = \det(A - \lambda I) \]

so $p(0) = \det(A - 0 \cdot I) = \det A$

so $\det(A) = 0$ and $A$ is singular.

Similar matrices

Example of similar matrices:

\[
A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 17 & 6 \\ -35 & -12 \end{pmatrix}
\]
Compute \[ p_A(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} 1 - \lambda & 1 \\ -2 & 4 - \lambda \end{pmatrix} \]

\[(1 - \lambda)(4 - \lambda) + 2 = \lambda^2 - 5\lambda + 6\]

\[ p_B(\lambda) = \det(B - \lambda I) = \det \begin{pmatrix} 17 - \lambda & 6 \\ -25 & -12 - \lambda \end{pmatrix} \]

\[(17 - \lambda)(-12 - \lambda) + 210 = \lambda^2 - 5\lambda + 6\]

So \[ p_B(\lambda) = p_A(\lambda) \]

Let \[ Q = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} \]

Find \[ Q^{-1} = \frac{1}{\det(Q)} \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} \]

Now compute \[ Q^{-1}AQ \]

\[
\begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} -16 & -6 \\ -35 & -12 \end{pmatrix} = B
\]

So

\[ Q^{-1}AQ = B \]
We say that matrices $A$ and $B$ are similar if we can find a non-singular matrix $P$ such that

$$P^{-1}AP = B.$$ 

Fact: similar matrices have the same characteristic polynomial, and so the same eigenvalues.

Proof of this fact:

Given $B = P^{-1}AP$

$$p_B(\lambda) = \det(B - \lambda I) = \det(P^{-1}AP - \lambda I)$$

$$= \det(P^{-1}(AP - \lambda I P))$$

$$= \det(P^{-1}(A - \lambda I) P)$$

$$= \det(P^{-1}) \det(A - \lambda I) \det(P)$$

$$= \det(P^{-1}) \det(A - \lambda I) \det(P)$$

$$= P_A(\lambda) \det(P^{-1}) \det(P)$$
Eigenvalues of a diagonal matrix

\[ A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \]

\[ \det(A - \lambda I) = (2 - \lambda)(-1 - \lambda)(3 - \lambda) = 0 \]

\[ \Rightarrow \lambda = 2, \lambda = -1, \lambda = 3 \]

Same for triangular:

\[ B = \begin{pmatrix} 3 & 0 & 0 \\ 4 & -5 & 0 \\ 5 & 6 & 1 \end{pmatrix} \]

\[ \det(B - \lambda I) = (3 - \lambda)(-5 - \lambda)(1 - \lambda) = 0 \]

\[ \Rightarrow \lambda = 3, \lambda = -5, \lambda = 1. \]

So the eigenvalues of triangular (or diagonal) matrices are just the diagonal entries.
So diagonal matrices are easy to work with.

Another useful property of diagonal matrices: taking powers is easy.

\[
A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}
\]

\[
A^2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix}
\]

In general: \[A^n = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix}\]

Diagonalization (sect. 8.2)

We say that a matrix A is diagonalizable if we can turn it into a diagonal matrix by multiplying it on the right by some P and on the left by \(P^{-1}\).

So A is diagonalizable if it is similar to a diagonal matrix.

Note: suppose we have
\[ p^{-1} A p = D \text{ (diagonal)} \]

For example, \[ p^{-1} A p = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \]

Then \[ (p^{-1} A p)^2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^2 = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix} \]

\[ p^{-1} A p p^{-1} A p = p^{-1} A I A p = p^{-1} A A p = p^{-1} A^2 p \]

So \[ p^{-1} A^2 p = \begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix} \]

So on, we can do same for any exponent:

\[ p^{-1} A^n p = \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} \]

and \[ A^n = p \begin{pmatrix} 2^n & 0 \\ 0 & 3^n \end{pmatrix} p^{-1} \]

p. 420 #12 sect. 8.1. Check all answers for this problem. Turn it in on Mon.

Go to the talk by Prof. David Cox tomorrow at 10:50 room 320, write a summary (30 pts).