Day 21
Wed Jun 21
Now compute the determinant by reducing the matrix to triangular form.

\[ A = \begin{pmatrix} 1 & t_0 & t_0^3 \\ -1 & t_1 & t_1^3 \\ 1 & t_2 & t_2^3 \end{pmatrix} \]

expand along column 3:

\[ t_0^3 \det \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \end{pmatrix} - t_1^3 \det \begin{pmatrix} 1 & t_0 \\ 1 & t_2 \end{pmatrix} + t_2^3 \det \begin{pmatrix} 1 & t_0 \\ 1 & t_1 \end{pmatrix} \]

\[ t_0^3 (t_2 - t_1) - t_1^3 (t_2 - t_0) + t_2^3 (t_1 - t_0) \]

\[ t_0^3 (t_2 - t_1) + t_1^3 (t_0 - t_2) + t_2^3 (t_1 - t_0) \]

\[ -(a - b) = b - a \]

\[ -(-b) = b \]

Now compute the determinant by reducing the matrix to triangular form.
\[
\begin{vmatrix}
1 & t_0 & t_0^3 \\
0 & t_1 - t_0 & (t_1 - t_0)(t_1^2 + t_1 t_0 + t_0^2) \\
0 & t_2 - t_0 & (t_2 - t_0)(t_2^2 + t_2 t_0 + t_0^2)
\end{vmatrix}
\]

\[
= (t_1 - t_0)(t_2 - t_0) \det
\begin{vmatrix}
1 & t_0 & t_0^3 \\
0 & 1 & t_1 + t_1 t_0 + t_0^2 \\
0 & 1 & t_2 + t_2 t_0 + t_0^2
\end{vmatrix}
\]

\[
= (t_1 - t_0)(t_2 - t_0) \det
\begin{vmatrix}
1 & t_0 & t_0^3 \\
0 & 1 & t_1 + t_1 t_0 + t_0^2 \\
0 & 0 & t_2 - t_1^2 + t_2 t_0 - t_1 t_0
\end{vmatrix}
\]

Note: \[
\begin{align*}
t_2^2 - t_1^2 + t_2 t_0 - t_1 t_0 &= (t_2 - t_1)(t_2 + t_1) + t_0(t_2 - t_1) \\
&= (t_2 - t_1)\left(t_2 + t_1 + t_0\right)
\end{align*}
\]
\[ (t_1 - t_0)(t_2 - t_0) \, dt = \begin{pmatrix} 1 & t_0 & t_0^3 \\ 0 & 1 & t_1 + t_1t_0 + t_0^2 \\ 0 & 0 & (t_2 - t_1)(t_0 + t_1 + t_2) \end{pmatrix} \]

\[ = (t_1 - t_0)(t_2 - t_0)(t_2 - t_1)(t_0 + t_1 + t_2) \]

So,

\[ t_0^3(t_2 - t_1) + t_1^3(t_0 - t_2) + t_2^3(t_1 - t_0) \]

\[ = (t_1 - t_0)(t_2 - t_0)(t_2 - t_1)(t_0 + t_1 + t_2) \]
Standard unit vectors in 3 dimensions:

\[ \hat{i} = (1, 0, 0) \quad \hat{j} = (0, 1, 0) \quad \hat{k} = (0, 0, 1) \]

\[ \|\hat{i}\| = 1 \quad \|\hat{j}\| = 1 \quad \|\hat{k}\| = 1 \]

\[ \hat{i} \cdot \hat{j} = 0 \quad \hat{i} \cdot \hat{k} = 0 \quad \hat{j} \cdot \hat{k} = 0 \]

Any vector in \( \mathbb{R}^3 \) can be written as a linear combinations of \( \hat{i}, \hat{j}, \hat{k} \):

\[ (3, -2, 1) = 3\hat{i} - 2\hat{j} + \hat{k} \]

Triangular inequality
So we get:

\[
|u + v| \leq |u| + |v|
\]

We can check it for specific examples:

\(u = (5, 3, 2), \ v = (1, 6, 0)\) Check the triangular inequality.

\[
|u| = \sqrt{5^2 + 3^2 + 2^2} = \sqrt{38} \quad |v| = \sqrt{1^2 + 6^2 + 0^2} = \sqrt{37}
\]

\(u + v = (6, 9, 2)\)

\[
|u + v| = \sqrt{6^2 + 9^2 + 2^2} = \sqrt{121} = 11
\]

\[11 \leq \sqrt{38} + \sqrt{37} \quad \text{TRUE.}\]

Sect. 4.3 Transformations from \(\mathbb{R}^m\) to \(\mathbb{R}^n\)

We write \(L : \mathbb{R}^m \rightarrow \mathbb{R}^n\) to mean that \(L\) is a function with domain \(\mathbb{R}^m\) and range \(\mathbb{R}^n\).

Ex:

\(L : \mathbb{R} \rightarrow \mathbb{R}, \quad L(x) = e^x\)
Note: here by range we mean the "type" of output we get (a number, or a vector in $\mathbb{R}^2$ or a vector in $\mathbb{R}^3$ etc.), so it is not necessary that *each* element in the range is actually an output of the function.

Ex: $L: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$L \left( x_1, x_2 \right) = \left( 2x_1 - 4, x_2 + x_2, x_1 - x_2 \right)$$
is an example of a function with domain $\mathbb{R}^2$ and range $\mathbb{R}^3$

A function $L: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is called linear if the following two conditions are satisfied:

1. $L \left( u + v \right) = L(u) + L(v)$
   for all vectors $u, v$ in $\mathbb{R}^m$

2. $L \left( t v \right) = t L(v)$
   for all vectors $v$ and numbers $t$. 

Note: most functions studied in Calculus are not linear. For example, $f(x) = e^x$ is not linear.

The only linear functions in one variable are straight lines through the origin:

$$f(x) = mx$$

$$f(x+y) = m(x+y) = mx + my = f(x) + f(y)$$

$$f(tx) = m(tx) = t(mx) = tf(x)$$

In general, to prove that a function is linear we need to check that both conditions (1) and (2) are true.