Day 18
Wed Jun 14
Vandermonde matrix and polynomial interpolation

The polynomial interpolation problem is:

Given a set of \( n + 1 \) points,

\[(t_0, y_0), (t_1, y_1), (t_2, y_2), \ldots, (t_n, y_n)\]

Find a polynomial of degree at most \( n \) that goes through all of them.

If \( n=1 \), we have two points, and a polynomial of degree 1 is a straight line.

If \( n=2 \), we have three points, and we look for a quadratic polynomial.

Case \( n=2 \): we look for a polynomial of form

\[ p(t) = c_0 + c_1 t + c_2 t^2 \]

such that

\[ p(t_0) = y_0 \quad p(t_1) = y_1 \quad p(t_2) = y_2 \]
So: \[ c_0 + c_1 t_0 + c_2 t_0^2 = \gamma_0 \]
\[ c_0 + c_1 t_1 + c_2 t_1^2 = \gamma_1 \]
\[ c_0 + c_1 t_2 + c_2 t_2^2 = \gamma_2 \]
The coefficient matrix is called the Vandermonde matrix

\[
A = \begin{pmatrix}
1 & t_0 & t_0^2 \\
1 & t_1 & t_1^2 \\
1 & t_2 & t_2^2
\end{pmatrix}
\]

The augmented matrix is

\[
\begin{pmatrix}
1 & t_0 & t_0^2 & y_0 \\
1 & t_1 & t_1^2 & y_1 \\
1 & t_2 & t_2^2 & y_2
\end{pmatrix}
\]

The vector of variables is

\[
\begin{pmatrix}
c_0 \\
c_1 \\
c_2
\end{pmatrix}
\]

and

\[
A \begin{pmatrix}
c_0 \\
c_1 \\
c_2
\end{pmatrix} = \begin{pmatrix} y_0 \\
y_1 \\
y_2 \end{pmatrix}
\]

So if the Vandermonde matrix A is non-singular, then we can solve the problem,

\[
\begin{pmatrix}
c_0 \\
c_1 \\
c_2
\end{pmatrix} = A^{-1} \begin{pmatrix} y_0 \\
y_1 \\
y_2 \end{pmatrix}
\]
We now find the determinant of the Vandermonde matrix to show that it is always non-singular, provided the given values of $t$ are all distinct.

\[
\det \begin{pmatrix}
1 & t_0 & t_0^2 \\
1 & t_1 & t_1^2 \\
1 & t_2 & t_2^2 \\
\end{pmatrix} = R_2 - R_1
\]

\[
= \det \begin{pmatrix}
1 & t_0 & t_0^2 \\
0 & t_1 - t_0 & t_1^2 - t_0^2 \\
0 & t_2 - t_0 & t_2^2 - t_0^2 \\
\end{pmatrix}
\]

\[
= \det \begin{pmatrix}
1 & t_0 & t_0^2 \\
0 & t_1 - t_0 & (t_1 - t_0)(t_1 + t_0) \\
0 & t_2 - t_0 & (t_2 - t_0)(t_2 + t_0) \\
\end{pmatrix}
\]

\[
(t_1 - t_0)(t_2 - t_0) \det \begin{pmatrix}
1 & t_0 & t_0^2 \\
0 & 1 & t_1 + t_0 \\
0 & 1 & t_2 + t_0 \\
\end{pmatrix} = R_3 - R_2
\]
We see that Det(A) is not zero because the t values are all distinct.

Note: using the diagonal method, we get:

\[(t_1 - t_0)(t_2 - t_0) \begin{pmatrix} 1 & t_0 & t_0^2 \\ 0 & 1 & t_1 + t_0 \\ 0 & 0 & t_2 - t_1 \end{pmatrix}
\]

\[= (t_1 - t_0)(t_2 - t_0)(t_2 - t_1) \neq 0\]

HW Factor the expression

\[t_0^3 (t_2 - t_1) + t_1^3 (t_0 - t_2) + t_2^3 (t_1 - t_0)\]
Hint: compute the determinant of

\[
\begin{pmatrix}
1 & t_0 & t_0^3 \\
1 & t_1 & t_1^3 \\
1 & t_2 & t_2^3 \\
\end{pmatrix}
\]

by doing a minor expansion along column 3, then compute it again by reducing it to triangular form.

Chapter 4 Vectors in $\mathbb{R}^n$

A vector in $\mathbb{R}^2$ can be viewed as an arrow. If not otherwise specified, the tail of the arrow is at the origin.

\[ v = (2, -1) \quad \text{or} \quad v = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \]
Now sketch the vector with tail at (1,3) and head at (5,6)

Now move the vector and put its tail at the origin.

In general, if a vector has head at \((x_1, y_1)\) and tail at \((x_o, y_o)\) then the same vector placed with the tail at the origin has the head at \((x_1 - x_o, y_1 - y_o)\)

We do not consider vectors with same length and direction to be different if their tail is at different places.

Length of a vector: Given a vector \(\mathbf{v} = (v_1, v_2)\)
the length is given by the Pythagorean theorem:
\[ \|v\| = \sqrt{v_1^2 + v_2^2} \]

Parallel vectors: Two vectors are parallel if one is a multiple of the other.

\[ v = (4, 3) \quad u = (8, 6) = 2v \]
Area of a triangle

Suppose we are given just the three vertices of a triangle, as points in $\mathbb{R}^2$.

Then the area is given by

$$\text{Area} = \frac{1}{2} \left| \begin{array}{ccc} -1 & 4 & 1 \\ 3 & 1 & 1 \\ 2 & 6 & 1 \end{array} \right|$$