Day 6
Wed May 17

Quiz 1 on Monday.
Linear combinations

A linear combination of row vectors (all of the same size)

\[ v_1, v_2, \ldots, v_n \]

is any sum of form

\[ a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \]

where \( a_1, a_2, \ldots, a_n \) are numbers.

Same for matrices: Given matrices

\[ A_1, A_2, \ldots, A_n \]

all of same size, a linear combination is a sum of form

\[ a_1 A_1 + a_2 A_2 + \ldots + a_n A_n \]

The main diagonal of a *square* matrix is the list of entries going from the top left corner to the bottom right corner, as shown:
We say that a matrix is diagonal if all entries that are not on the main diagonal are zero.

Ex:

\[
\begin{pmatrix}
 a_{11} & a_{12} & a_{13} \\
 a_{21} & a_{22} & a_{23} \\
 a_{31} & a_{32} & a_{33}
\end{pmatrix}
\]

is diagonal

\[
\begin{pmatrix}
 3 & 0 & 0 \\
 0 & -1 & 0 \\
 0 & 0 & 1
\end{pmatrix}
\]

is diagonal

\[
\begin{pmatrix}
 2 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 7
\end{pmatrix}
\]

is diagonal

Matrix multiplication

The multiplication of two matrices $A$ and $B$ is defined *only* if the number of columns of $A$ is the same as the number of rows of $B$. 
Ex: $\begin{bmatrix} A & B & C \\ 2 & 3 & 4 & 2 \end{bmatrix}$

$AB$ is defined $AC$ is not

$\begin{bmatrix} 2 & 3 & 2 & 4 \\ 3 & 4 & 3 \end{bmatrix}$

$BC$ is defined $CA$ defined

$\begin{bmatrix} 3 & 4 & 2 & 3 \end{bmatrix}$

We see that matrix multiplication in general will not be commutative: $AB$ and $BA$ are not the same.

We now define the multiplication.

$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}$

$(AB)_{11} = (\text{Row 1 of } A) \cdot (\text{Column 1 of } B)$

$= (1)(-2) + (2)(4) + (-1)(2) = -2 + 8 - 2 = 4$

$(AB)_{12} = (\text{Row 1 of } A) \cdot (\text{Column 2 of } B)$
\[ (1)(5) + (2)(-3) + (-11)(1) = -2 \]

\[
(AB)_{21} = (\text{Row 2 of } A) \cdot (\text{Column 1 of } B) = 6
\]

\[
(AB)_{22} = (\text{Row 2 of } A) \cdot (\text{Column 2 of } B) = 16
\]

\[
AB = \begin{pmatrix}
4 & -2 \\
6 & 16
\end{pmatrix}
\]

Now do BA:

\[
BA = \begin{pmatrix}
-2 & 5 \\
4 & -3 \\
2 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 & -1 \\
3 & 1 & 4
\end{pmatrix}
= \begin{pmatrix}
13 & 1 & 22 \\
-5 & 5 & -16 \\
5 & 5 & 2
\end{pmatrix}
\]

Note: In general, the size of the product $AB$ is $m \times p$

If $A$ is $m \times n$, then $A^T$ is $n \times m$. So if $A$, $B$ are not defined. But $A^T B$ is defined.
Note: \( A^2 = A \cdot A \) is only defined when \( A \) is a square matrix, \( A_{n \times n} \)

Connection between matrix multiplication and linear systems

Consider the linear system

\[
\begin{align*}
    x_1 + 2x_2 - x_3 &= 2 \\
    3x_1 + 4x_3 &= 5
\end{align*}
\]

The coefficient matrix is

\[
A = \begin{pmatrix}
    1 & 2 & -1 \\
    3 & 0 & 4
\end{pmatrix}
\]

Column vector of the constants is

\[
b = \begin{pmatrix}
    2 \\
    5
\end{pmatrix}
\]

Column vector of the variables is

\[
x = \begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix}
\]

Note that we can multiply \( Ax \)
So we can write the linear system in a compact way as:
\[ A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 - x_3 \\ 3x_1 + 4x_3 \end{pmatrix} \]

If these were numbers, we would solve for \( x \):
\[ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}b \]
So if we can find the meaning of \( A^{-1} \) for a matrix, then we can solve the system in the same way.

HW p. 34 #1, 3, 5, 7, 9, 11, 15, 19, 21

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